# Exam Calculus 1

7 november 2011, 14.00-17.00.

This exam has 9 problems. Each problem is worth 1 point; more details can be found below. Write on each page your name and student number, and on the first page your seminar group. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by arguments and/or work. Success.

1. (a) Formulate the principle of mathematical induction.

   (b) Prove that if $n \geq 1$ is a positive integer, then

   $$n^3 + 2n$$

   is divisible by 3.

2. Find all (complex) solutions of

   $$(z+1)^3 = -2 - 2\mathrm{i}$$

   and plot them in the complex plane.

3. (a) Let $f(x)$ be a function defined on some open interval that contains the number $a$ (except possibly at $a$ itself). Give the $\epsilon$-$\delta$-definition of

   $$\lim_{x \to a} f(x) = L$$

   (b) Prove that (using the $\epsilon$-$\delta$-definition)

   $$\lim_{x \to 0} x \sin \frac{1}{x} = 0$$

4. We assume that $f$ is a differentiable function with $f(x) \neq 0$.
   Fill in the missing pieces:

   $$\frac{d}{dx} \frac{1}{f(x)} = \lim_{h \to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \cdots = \frac{-f'(x)}{(f(x))^2}$$

5. Find the derivative of

   $$\mathrm{e}^{(2x-1)^2} + \int_{2x-1}^{x^2+1} \mathrm{e}^{t^2}\, dt$$

6. Newton's law of Gravitation says that the magnitude $F$ of the force exerted by the earth on a body with unit mass at a distance $r$ from the center of the earth is given by

$$F(r) = \begin{cases} GMr/R^3 & \text{if } r < R \\ \\ GM/r^2 & \text{if } r \geq R \end{cases}$$

where $R$ denotes the radius of the earth, $M$ is the mass of the earth and $G$ is the gravitational constant. (Note: $R$, $M$ and $G$ are constant).

(a) Is the function $F(r)$ continuous?

(b) Is $F$ differentiable at $r = R$?

7. The function $f$ is given by $f(x) = x^x$ where $x > 0$.

(a) Show that

$$\lim_{x \to 0^+} f(x) = 1$$

(b) Find $f'(x)$ for $x > 0$.

(c) Evaluate

$$\lim_{x \to 0^+} \frac{f(x) - 1}{x}$$

8. (a) Evaluate

$$\int \frac{\cos \sqrt{2x}}{\sqrt{x}}\, dx$$

(b) Idem

$$\int_0^1 \sqrt{x} \ln x \, dx$$

9. Take $x > 0$. Find the solution $y(x)$ of the differential equation

$$x^2 y' + xy = 1$$

that satisfies $y(1) = 2$.

Maximum score:

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1a | 0.5 | 2 | 1.0 | 3a | 0.5 | 4 | 1.0 | 5 | 1.0 | 6a | 0.5 | 7a | 0.4 | 8a | 0.5 | 9 | 1.0 |
| b | 0.5 | | | b | 0.5 | | | | | b | 0.5 | b | 0.3 | b | 0.5 | | |
| | | | | | | | | | | | | c | 0.3 | | | | |